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1999 J. Phys. A: Math. Gen. 32 2697

(http://iopscience.iop.org/0305-4470/32/14/010)

J. Phys. A: Math. Gen. **32** (1999) 2697–2703. Printed in the UK PII: S0305-4470(99)98493-7

# High-frequency focus wavemodes in uniaxial anisotropic dielectrics

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Received 19 October 1998

**Abstract.** We investigate in the high-frequency limit the propagation of focus wavemodes in uniaxial anisotropic dielectrics. We prove that, as for harmonic plane waves, two sets (ordinary and extraordinary) of focus wavemodes can propagate in this medium. We discuss some possible consequences of this result.

## 1. Introduction

The focus wavemode (FWM) solutions of the wave equation, discovered independently by Brittingham [1] and Kiselev [2], are in fact a particular class of more general solutions known to Courant–Hilbert as distortion-free progressing waves [3]. Many works have been devoted to FWM propagation in isotropic media [4–6]. However, because of the important applications of anisotropic media in the field of electromagnetic waves, for instance in plasmas, ferrite devices, travelling wave tubes, laser systems and so on, and because, hopefully, FWM beams could be used to transport information over large distances, it is necessary to investigate whether and how FWM propagation takes place in anisotropic media. We start here by considering the simplest case of a uniaxial anisotropic dielectric.

Choosing coordinates along the principal axes of the permittivity tensor, we define a uniaxial anisotropic dielectric by the constitutive relations

$$D_{x,y} = \varepsilon E_{x,y} \qquad D_z = \eta E_z \qquad \boldsymbol{B} = \mu \boldsymbol{H} \tag{1}$$

in which $\varepsilon$ and $\eta$ depend on the frequency $\omega$ while $\mu$ is a constant scalar. We look for the solutions of Maxwell equations in the form [3]

$$H_j = a_j \exp(\mathrm{i}\omega\chi) \qquad D_j = b_j \exp(\mathrm{i}\omega\chi) \qquad j = 1, 2, 3 \tag{2}$$

in which $\chi$ is a solution of the characteristic equation with $n^2 = \varepsilon\mu$, $m^2 = \eta\mu$

$$m^{-2}\{(\partial_x\chi)^2 + (\partial_y\chi)^2\} + n^{-2}(\partial_z\chi)^2 - c^{-2}(\partial_t\chi)^2 = 0 \tag{3}$$

while $a_j$ and $b_j$ are amplitudes to be determined. In these expressions, the index $j$ takes values $1, 2, 3$, corresponding to the coordinates $x$, $y$, $z$, respectively.

We first consider the propagation of TE and TM focus wavemodes when the electromagnetic field does not depend on the coordinate $y$.

0305-4470/99/142697+07$19.50  

## 2. Propagation of TM and TE focus wavemodes

For a TM field, Maxwell equations reduce to

$$\partial_z H_y = -c^{-1}\partial_t D_x \qquad \partial_x H_y = c^{-1}\partial_t D_z \tag{4a}$$

$$\partial_z E_x - \partial_x E_z = -\mu c^{-1}\partial_t H_y. \tag{4b}$$

We assume that the component $H_y$ is the scalar focus wavemode [1, 2]

$$H_y = g^{-1/2}\psi \qquad \psi = \exp(\mathrm{i}\omega\chi) \tag{5}$$

in which $\chi$ is the solution of the two-dimensional (2D) characteristic equation

$$m^{-2}(\partial_x\chi)^2 + n^{-2}(\partial_z\chi)^2 - c^{-2}(\partial_t\chi)^2 = 0 \tag{6a}$$

while $g$ is determined so that $H_y$ is a solution of the 2D wave equation

$$(m^{-2}\partial_x^2 + n^{-2}\partial_z^2 - c^{-2}\partial_t^2)H_y = 0. \tag{6b}$$

In addition, as previously stated, we are interested in the high-frequency solutions of equations (4), that is, we assume $\omega$ (which is in fact a wavenumber) is very large so that, for any first derivative of an expression $A\psi$, we may neglect $(\partial A)\psi$ with respect to $\mathrm{i}\omega A(\partial\chi)_\psi$ and write

$$\partial(A\psi) = \mathrm{i}\omega\{A\partial\chi\psi + 0(\omega^{-1})\}. \tag{7}$$

### *2.1. Propagation along the symmetry axis of the anisotropic dielectric*

One checks easily that in this case equations (6$a$, $b$) are satisfied with

$$\chi = ct - nz - m^2x^2g^{-1} \qquad g = ct + a + nz \tag{8}$$

and a simple calculation gives

$$\partial_x\chi = -2mxg^{-1} \qquad \partial_z\chi = -n(1 - m^2x^2g^{-2}) \qquad c^{-1}\partial_t\chi = 1 + m^2x^2g^{-2}. \tag{9}$$

Using (7) (that is, neglecting $\partial g^{-1/2}$) and (9), we get

$$\partial_x H_y = -\mathrm{i}\omega\{2m^2xg^{-1}H_y + 0(\omega^{-1})\}$$

$$\partial_z H_y = -\mathrm{i}\omega\{n(1 - m^2x^2g^{-2})H_y + 0(\omega^{-1})\} \tag{10a}$$

$$c^{-1}\partial_t H_y = \mathrm{i}\omega\{(1 + m^2x^2g^{-2})H_y + 0(\omega^{-1})). \tag{10b}$$

We now look for the solutions of equations (4$a$) in the form $D_{x,z} = f_{x,z}\psi$ in which the amplitudes $f_{x,z}$ are to be determined. Using (7) (with $A = f_{x,z}$) and (9) gives

$$c^{-1}\partial_t D_{x,z} = \mathrm{i}\omega\{(1 + m^2x^2g^{-2})f_{x,z}y + 0(\omega^{-1})\}. \tag{11}$$

Substituting (10$a$) and (11) into (4$a$) we get at once

$$f_x = ng^{-1/2}(g^2 - m^2x^2)(g^2 + m^2x^2)^{-1}$$

$$f_z = -2m^2xg^{1/2}(g^2 + m^2x^2)^{-1} \tag{12}$$

so that with $\varepsilon_x = \varepsilon$, $\varepsilon_z = \eta$, the $x$, $z$-components of the electric field are

$$E_{x,z} = (\varepsilon_{x,z})^{-1}f_{x,z}\psi = b_{x,z}H_y \tag{13}$$

$$b_x = (\mu/\varepsilon)^{1/2}(g^2 - m^2x^2)(g^2 + m^2x^2)^{-1}$$

$$b_z = -2\mu gz(g^2 + m^2x^2)^{-1}. \tag{13a}$$

One checks easily that with (5) and (13), equation (4$b$) is satisfied. We have indeed

$$\partial_z E_x = \mathrm{i}\omega\{b_x\partial_z\chi H_y + 0(\omega^{-1})\} \tag{14a}$$

$$= -\mathrm{i}\omega\{n(1 - m^2x^2g^{-2})b_xH_y + 0(\omega^{-1})\} \tag{14b}$$

and using (13*a*)

$$\partial_z E_x = -\mathrm{i}\omega\{\mu g^2(g^2 - m^2x^2)^2(g^2 + m^2x^2)^{-1} H_y + 0(\omega^{-1})\}. \qquad (14c)$$

Similarly

$$\partial_x E_z = \mathrm{i}\omega\{b_z \partial_x c H_y + 0(\omega^{-1})\} \qquad (15a)$$

$$= -\mathrm{i}\omega\{2m^2 x g^{-1} b_z H_y + 0(\omega^{-1})\} \qquad (15b)$$

$$\partial_x E_z = \mathrm{i}\omega\{4\mu m^2 x^2 (g^2 + m^2x^2)^{-1} H_y + 0(\omega^{-1})\}. \qquad (15c)$$

From (10*b*), (14*c*) and (15*c*) we get equation (4*b*).

### 2.2. Propagation in an arbitrary direction

We now assume that the focus wavemode propagates in a direction making the angle $u$ with the $z$-axis. In this case

$$\chi = ct - Z - g^{-1}X^2 \qquad g = ct + a + Z \qquad (16)$$

$$Z = nz\cos u + mx\sin u \qquad X = mx\cos u - nz\sin u. \qquad (16a)$$

Then

$$\partial_x \chi = -m[\sin u(1 - g^{-2}X^2) + 2g^{-1}\cos uX]$$

$$\partial_z \chi = -n[\cos u(1 - g^{-2}X^2) - 2g^{-1}\sin uX] \qquad (17a)$$

$$c^{-1}\partial_t c = 1 + g^{-2}X^2. \qquad (17b)$$

So, still using (7)

$$\partial_x H_y = -\mathrm{i}\omega\{m[\sin u(1 - g^{-2}X^2) + 2g^{-1}\cos uX]H_y + 0(\omega^{-1})\}$$

$$\partial_z H_y = -\mathrm{i}\omega\{n[\cos u(1 - g^{-2}X^2) - 2g^{-1}\sin uX]H_y + 0(\omega^{-1})\} \qquad (18a)$$

$$c^{-1}\partial_t H_y = \mathrm{i}\omega\{(1 + g^{-2}X^2)H_y + 0(\omega^{-1})\}. \qquad (18b)$$

Similarly to (11) with $X^2$ taking the place of $m^2x^2$

$$c^{-1}\partial_t D_{x,z} = \mathrm{i}\omega\{(1 + g^{-2}X^2) f_{x,z}\psi + 0(\omega^{-1})\}. \qquad (19)$$

Substituting (18*a*) and (19) into (4*a*) we easily get

$$f_x = ng^{-1/2}(g^2 + X^2)^{-1}[\cos u(g^2 - X^2) - 2\sin ugX]$$

$$f_z = -mg^{-1/2}(g^2 + X^2)^{-1}[\sin u(g^2 - X^2) + 2\cos ugX] \qquad (20)$$

and the amplitudes $b_{x,z}$ in (13) become

$$b_x = (\mu/\varepsilon)^{1/2}(g^2 + X^2)^{-1}[\cos u(g^2 - X^2) - 2\sin ugX]$$

$$b_z = (\mu/\eta)^{1/2}(g^2 + X^2)^{-1}[\sin u(g^2 - X^2) + 2\cos ugX]. \qquad (21)$$

In this case also one easily checks that equation (4*b*) is satisfied since equations (14*a*), (15*a*) become

$$\partial_z E_x = \mathrm{i}\omega\{ng^{-2}[\cos u(g^2 - X^2) - 2\sin ugX]b_x H_y + 0(\omega^{-1})\}$$

$$\partial_x E_z = -\mathrm{i}\omega\{mg^{-2}[\sin u(g^2 - X^2) + 2\cos ugX]b_z H_y + 0(\omega^{-1})\} \qquad (22a)$$

and using (21) ($0(\omega^{-1})$ understood)

$$\partial_z E_x = -\mathrm{i}\omega\{\mu g^{-2}(g^2 + X^2)^{-1}[\cos^2 u(g^2 - X^2)^2 + 4\sin^2 ug^2X^2$$
$$-2\sin 2ugX(g^2 - X^2)]H_y\}$$

$$\partial_x E_z = \mathrm{i}\omega\{\mu g^{-2}(g^2 + X^2)^{-1}[\sin^2 u(g^2 - X^2)^2 + 4\cos^2 ug^2X^2$$
$$+2\sin 2ugX(g^2 - X^2)]H_y\} \qquad (22b)$$

so that $\mu^{-1}(\partial_x E_z - \partial_z E_x) = (18b)$ which is equation (4*b*).

To sum up, the electromagnetic TM focus wavemode propagating in the $u$-direction of the uniaxial anisotropic dielectric is

$$E_{x,y} = b_{x,y} H_y \qquad H_y = g^{-1} \exp(\mathrm{i}\omega\chi) \tag{23}$$

with $b_{x,z}$ given by (21), $g$ and $\chi$ by (16).

**Remark.** The situation is different for TE focus wavemodes since Maxwell equations are

$$\partial_z D_y = n^2 c^{-1} \partial_t H_x \qquad \partial_x D_y = n^2 c^{-1} \partial_t H_z \tag{24$a$}$$

$$\partial_z H_x - \partial_x H_z = c^{-1} \partial_t D_y. \tag{24$b$}$$

We look for the solutions of equations (24) in the form $D_y = g^{-1} \exp(\mathrm{i}\omega\chi^\circ)$ with $g$, $\chi^\circ$ also given by (13) but with $m = n$ in (13$a$) while $H_{x,z} = a_{x,z} D_y$; still using (7), one could easily obtain $a_{x,z}$. Anticipating on the results to be proved in the next section, one could name TM and TE focus wavemodes extraordinary and ordinary waves, respectively.

## 3. Propagation of an arbitrary focus wavemode

We write Maxwell equations

$$\begin{aligned}
&\partial_y H_z - \partial_z H_y = c^{-1} \partial_t D_x \qquad && m^{-2} \partial_y D_z - n^{-2} \partial_z D_y = -c^{-1} \partial_t H_x \\
&\partial_z H_x - \partial_x H_z = c^{-1} \partial_t D_y \qquad && n^{-2} \partial_z D_x - m^{-2} \partial_x D_z = -c^{-1} \partial_t H_y \\
&\partial_x H_y - \partial_y H_x = c^{-1} \partial_t D_z \qquad && n^{-2} \partial_x D_y - n^{-2} \partial_y D_x = -c^{-1} \partial_t H_x.
\end{aligned} \tag{25}$$

As stated in the introduction, we look for the solutions of equations (25) in the form (2). We still use the high-frequency approximation (7), but from now on for simplification we no longer write $0(\omega^{-1})$ and we introduce the functions

$$w_j = \partial_j \chi / c^{-1} \partial_t \chi \qquad \partial_j = \partial/\partial x_j \qquad j = 1, 2, 3. \tag{26}$$

According to equation (3), they satisfy the relation

$$m^{-2}(w_x^2 + w_y^2) + n^{-2} w_z^2 - 1 = 0. \tag{26$a$}$$

Then, substituting (2) into (25), using (7) and (24), the Maxwell equations become

$$\begin{aligned}
&w_y a_z - w_z a_y - b_x = 0 \qquad && m^{-2} w_y b_z - n^{-2} w_z b_y + a_x = 0 \\
&w_z a_x - w_x a_z - b_y = 0 \qquad && n^{-2} w_z b_x - m^{-2} w_x b_z + a_y = 0 \\
&w_x a_y - w_y a_x - b_z = 0 \qquad && n^{-2} w_x b_y - n^{-2} w_y b_x + a_z = 0
\end{aligned} \tag{27}$$

which is an homogeneous system of six equations for the six unknowns $a_j$, $b_j$, with a non-trivial solution only if its determinant is zero. This determinant given in appendix A can be changed by elementary transformations into the equivalent determinant

$$M = \begin{bmatrix} -1 + n^{-2}(w_x^2 + w_z^2) & -n^{-2} w_x w_y & -m^{-2} w_z w_x \\ -n^{-2} w_x w_y & -1 + n^{-2}(w_x^2 + w_z^2) & -m^{-2} w_z w_y \\ -n^{-2} w_x w_z & -n^{-2} w_y w_z & -1 + m^{-2}(w_x^2 + w_z^2) \end{bmatrix}. \tag{28}$$

Using (26$a$), the element $M_{33}$ of (28) may be replaced by $-n^{-2} w_z^2$ so that

$$M = -n^{-2} w_z^2 \begin{bmatrix} -1 + n^{-2}(w_y^2 + w_z^2) & -n^{-2} w_y w_x & -m^{-2} w_x \\ -n^{-2} w_x w_y & -1 + n^{-2}(w_x^2 + w_z^2) & -m^{-2} w_y \\ w_x & w_y & 1 \end{bmatrix} \tag{29}$$

and we prove in appendix B that $M = 0$, so the system (26) always has a non-trivial solution. Let us, for instance, consider the following solution of equation (3)

$$\chi = ct - nz - m^2 r^2 g^{-1} \qquad g = ct + a + nz \qquad r^2 = x^2 + y^2. \tag{30}$$

According to (27) we may assume that one arbitrary component of the electromagnetic field (say $D_z$) is a scalar FWM solution of the wave equation

$$\{m^{-2}(\partial_x^2 + \partial_y^2) + n^{-2}\partial_z^2 - c^{-2}\partial_t^2\}D_z = 0. \tag{31}$$

Then [1, 2], taking into account (30)

$$D_z = g^{-1} \exp(\mathrm{i}\omega\chi) \tag{32}$$

(note that the amplitude of $D_z$ is $g^{-1}$ instead of $g^{-1/2}$ as in (5), the reason of this difference will be discussed elsewhere). We get from (30) in the high-frequency approximation

$$\begin{aligned} &\partial_x\chi = -2m^2xg^{-1} \qquad \partial_y\chi = -2m^2yg^{-1} \\ &\partial_z\chi = -n(1 - m^2r^2g^{-2}) \qquad c^{-1}\partial_t\chi = 1 + m^2r^2g^{-2} \end{aligned} \tag{33}$$

so that according to (26)

$$\begin{aligned} w_x &= -2m^2xg(g^2 + m^2r^2)^{-1} \\ w_y &= -2m^2yg(g^2 + m^2r^2)^{-1} \\ w_z &= -n(g^2 - m^2r^2)(g^2 + m^2r^2)^{-1}. \end{aligned} \tag{34}$$

Then, substituting (34) into (27) will supply $a_j, b_j$, in terms of the amplitude $a_3 = g^{-1}$ of $D_z$. We refrain from making this simple but rather long calculation.

Let us now inquire what happens to $\chi$ if we note that $\chi^\circ$ is a solution of the characteristic equation that one would meet in an isotropic medium

$$n^{-2}\{(\partial_x\chi^\circ)^2 + (\partial_y\chi^\circ)^2 + (\partial_z\chi^\circ)^2\} - c^{-2}(\partial_t\chi^\circ)^2 = 0 \tag{35}$$

so that relation (26*a*) (for simplification, we write $w$ for $w^\circ$) is changed into

$$n^{-2}(w_x^2 + w_y^2 + w_z^2) - 1 = 0. \tag{35a}$$

Then, using (35*a*) in the elements $M_{11}$ and $M_{22}$ of (28), the determinant $M$ becomes

$$\begin{aligned} M &= \begin{bmatrix} -n^{-2}w_x^2 & -n^{-2}w_xw_y & -m^{-2}w_zw_x \\ -n^{-2}w_xw_y & -n^{-2}w_y^2 & -m^{-2}w_zw_y \\ -n^{-2}w_xw_z & -n^{-2}w_yw_z & -1 + m^{-2}(w_x^2 + w_y^2) \end{bmatrix} \\ &= w_x^2w_y^2 \begin{bmatrix} n^{-2} & n^{-2} & m^{-2}w_z \\ n^{-2} & n^{-2} & m^{-2}w_z \\ -n^{-2}w_z & -n^{-2}w_z & -1 + m^{-2}(w_x^2 + w_y^2) \end{bmatrix} \end{aligned} \tag{36}$$

so that $M = 0$: in this case also, the system (27) has a non-trivial solution. This means that in addition to (2) a second set of focus wavemodes

$$H_j^\circ = a_j^\circ \exp(\mathrm{i}\omega\chi^\circ) \qquad D_j^\circ = b_j^\circ \exp(\mathrm{i}\omega\chi^\circ) \tag{37}$$

with $\chi^\circ$ a solution of the characteristic equation (35) can propagate in a uniaxial anisotropic dielectric. In agreement with the terminology used for harmonic plane waves [7–9], we name as extraordinary and ordinary focus wavemodes the solutions (2) and (37) whose particular case is supplied by the TM and TE components of the electromagnetic field.

## 4. Discussion

The similarity between harmonic plane wave (HPW) and high-frequency FWM propagation in uniaxial anisotropic dielectrics is striking but natural from a mathematical point of view since, as proved by Courant–Hilbert [3], HPW and FWM are undistorted progressing waves with the typical property that their phase is a solution of the characteristic equation. The exact form of the phase and of the attenuation factor (1 for HPW, $g^{-1}$ for FWM) is of secondary importance:

HPW and FWM propagate similarly in all space for $-\infty < t < \infty$. So, no physical wave can be distortion-free *stricto sensu* and one has to think of physical HPW and FWM as acceptable approximations in some bounded region of spacetime of mathematical HPW and FWM. Naturally the similarity between these waves suggests further works: reflection and refraction of FWM at surfaces of anisotropic dielectrics, the way that ordinary and extraordinary FWM are excited, the double refraction phenomenon and FWM propagation in biaxial anisotropic dielectrics.

All the results obtained in this work are valid only for high frequencies. Calculations become intricate at lower frequencies. for instance, for a TM-FWM we get instead of (10$a$) and (11)

$$\begin{aligned} \partial_z H_y &= -n\{g^{-1} + \mathrm{i}\omega(1 - m^2x^2g^{-2})\}H_y \\ c^{-1}\partial_t D_x &= \{c^{-1}\partial_t f_x + \mathrm{i}\omega(1 - m^2x^2g^{-2})\}\psi. \end{aligned} \tag{38}$$

Substituting (38) into the first equation (4$a$) gives the first-order differential equation

$$c^{-1}\partial_t f_x + \mathrm{i}\omega(1 + m^2x^2g^{-2})f_x = ng^{-3/2} + \mathrm{i}\omega n g^{-1/2}(1 - m^2x^2g^{-2}) \tag{39}$$

and looking for the solution of this equation is not an easy task.

In order for FWM to become a practical tool, for instance to transmit information at large distances as stated in the introduction, one must be able to generate approximate FWM. Many suggestions have been made in the past [10–12] but the most attractive one is that by Ziolkowski *et al* [13] who in addition have been able to check experimentally in acoustics the performances of their approximate FWM [14, 15]. No such experiment seems to have been made in electromagnetism.

## Appendix A

The determinant of the system (27) is

$$\begin{bmatrix} 0 & -w_z & w_y & -1 & 0 & 0 \\ w_z & 0 & -w_x & 0 & -1 & 0 \\ -w_y & w_x & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & -n^{-2}w_z & m^{-2}w_y \\ 0 & 1 & 0 & n^{-2}w_z & 0 & -m^{-2}w_x \\ 0 & 0 & 1 & -n^{-2}w_y & n^{-2}w_x & 0 \end{bmatrix}.$$

Subtracting the fourth line multiplied by $-w_z$ (respectively by $w_y$) from the second (respectively third) line changes into zero the first two elements of the first column. Using the same technique in the second and third columns transforms the previous determinant into the determinant (28).

## Appendix B

Leaving aside the factor $-n^{-2}w_z^2$ and expanding (29) with respect to the elements of the third line give, with $w^2 = w_x^2 + w_y^2 + w_z^2$

$$\begin{aligned} M = {} & [1 - n^{-2}(w_y^2 + w_z^2)][1 - n^{-2}(w_x^2 + w_z^2)] - n^{-4}w_x^2w_y^2 \\ & -w_y[m^{-2}w_y - m^{-2}n^{-2}w_yw^2] + w_x[-m^{-2}w_x + m^{-2}n^{-2}w_xw^2] \\ = {} & 1 - n^{-2}(w^2 + w_z^2) + n^{-4}w_z^2w^2 - m^{-2}(w_x^2 + w_y^2)(1 - n^2w^2). \end{aligned}$$

Using relation (26$a$) in the last term of this expression we get

$$M = 1 - n^{-2}(w^2 + w_z^2) + n^{-4}w_z^2w^2 + (n^{-2}w_z^2 - 1)(1 - n^{-2}w^2) = 0.$$

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